TMA4170 Fourier Analysis

Discrete Fourier analysis

Fourier series: $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x}, \qquad c_n = \int_0^1 f(x) e^{-2\pi i n x}\, dx$

Discrete Fourier transform: $F(k) = \sum_{n=0}^{N-1} c_n^N e^{2\pi i n \frac{k}{N}}, \qquad c_n^N = \frac{1}{N} \sum_{l=0}^{N-1} F(l) e^{-2\pi i n \frac{l}{N}}$
DFT

$$\boxed{F(k) := f\left(\tfrac{k}{N}\right) \implies \text{DFT} \xrightarrow[N\to\infty]{} \text{Fourier series}}$$

The group $\mathbb{Z}(N)$:

$$\mathcal{S}_N := \{ z \in \mathbb{C} : z^N = 1 \} = \{ 1, \zeta, \dots, \zeta^{N-1} \}, \quad \zeta := e^{\frac{2\pi i}{N}}$$

group under $\mathbb{C}$ - multiplication

$$\simeq \{ 0, 1, \dots, N-1 \} =: \mathbb{Z}(N)$$

group under addition mod N

isomorphic,

we identify elements $k \mapsto e^{2\pi i \frac{k}{N}}$

Fourier analysis on $\mathbb{Z}(N)$:

$V :=$ vectorspace of functions $F: \mathbb{Z}(N) \to \mathbb{C}$

$\simeq$ vectorspace of functions $f: \mathbb{S}_N \to \mathbb{C}$ $=: W$

Inner product / norm: $(F, G) = \sum_{n=1}^{N-1} F(k) \cdot \overline{G(k)}$, $\|F\|^2 = \sum_{k=1}^{N-1} |F(k)|^2$ $\quad V \sim \mathbb{C}^N$

Orthogonal basis: $e_0, \dots, e_{N-1}$ where $e_j(k) = e^{2\pi i j \frac{k}{N}}$ $\quad [\, (e_n, e_m) = N \delta_{nm} \,]$

DFT: $\quad F(k) = \sum_{j=0}^{N-1} c_j \, e_j(k)$, $\quad c_j = \frac{1}{N} \sum_{k=0}^{N-1} F(k) \, \overline{e_j(k)}$

Theorem: $F \in V$

(a) $F(k) = \sum_{k=0}^{N-1} c_n e^{2\pi i n \frac{k}{N}}$, $k \in \mathbb{Z}(N)$ $\quad$ Fourier-inversion

(b) $\sum_{n=0}^{N-1} |c_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |F(k)|^2$ $\quad$ Parseval